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ONE METHOD OF CONSTRUCTING POSITIVELY INVARIANT SETS FOR A LORENZ SYSTEM*

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An algorithm for constructing positively invariant sets containing separatrices which emerge from the singular zero saddle point O is proposed for a Lorenz system. The domains which contain bifurcation curves corresponding to the existence of separatrix loops of the saddle O in Lorenz's system are estimated. From the estimates obtained, in particular, it follows that when the Prandtl number approaches infinity the area in which there are no separatrix loops from the saddle O increases without limit. This fact was established earlier for some values of the parameters using numerical analysis /1/. In the algorithm proposed here we use the ideas discussed in /2-6/.

It is well-known /7/ that when there are three states of equilibrium Lorenz's system can be written in the form (μ, γ, A are positive numbers)

$$\sigma^{\cdot} = \eta, \quad \eta^{\cdot} = -\mu\eta - z\sigma - \varphi(\sigma), \quad z^{\cdot} = -Az - B\sigma\eta \qquad (1)$$
$$\varphi(\sigma) = -\sigma + \gamma\sigma^3, \quad B \in R^1$$

We shall consider the case when $B \geqslant 0$.

We shall take into consideration the continuous functions $P_0(\sigma) = {}^1/_2 B\sigma^2$, $Q_k(\sigma)$, $P_k(\sigma)$ $(k = 1, \ldots, N)$, which satisfy the equations

$$P_k(0) = 0 \qquad (2)$$
$$Q_k(0) = \rho; \quad Q_k'(0) > 0 \quad \text{for} \quad \rho = 0 \qquad (3)$$
$$\frac{dQ_k}{d\sigma} Q_k + \mu Q_k + \varphi(\sigma) - P_{k-1}(\sigma)\sigma = 0, \quad \forall \sigma \in (0, a_k) \qquad (4)$$
$$\frac{dP_k}{d\sigma} = B\sigma - \frac{AP_k}{Q_k(\sigma)}, \quad \forall \sigma \in (0, a_k) \qquad (5)$$

Here ρ is some non-negative number, and $[-a_k, a_k]$ is the maximum interval of definition of the solution $Q_k(\sigma)$ of Eq.(4) with the initial data (3). It is clear that the solution $P_k(\sigma)$ of Eq.(5) is also defined in this interval.

In some cases the equation $a_k = +\infty$ can be satisfied for a finite number k. However, we can show thatfor farily large values of k $a_k < +\infty$.

Let us further consider the continuous functions $p_0(\sigma) = {}^1/_2 B(a_N^2 - \sigma^2)$, $q_k(\sigma)$, $p_k(\sigma)$ $(k = 1, \ldots, M)$, which satisfy the equations

$$p_k(a_N) = 0 \qquad (6)$$
$$q_k(a_N) = 0, \quad q_k'(a_N) = -\infty \qquad (7)$$
$$\frac{dq_k}{d\sigma} q_k + \mu q_k + \varphi(\sigma) + p_{k-1}(\sigma)\sigma = 0, \quad \forall \sigma \in (\alpha_k, a_N) \qquad (8)$$
$$\frac{dp_k}{d\sigma} = -B\sigma - \frac{Ap_k}{q_k(\sigma)}, \quad \forall \sigma \in (\alpha_k, a_N) \qquad (9)$$

Here $[\alpha_k, a_N]$ is the maximum interval in which the solution $q_k(\sigma)$ of Eq.(8) with the initial data (7) is defined.

If $\alpha_k > 0$, we will supplement the definition of the function $p_k(\sigma)$ and $q_k(\sigma)$ in the folowing way:

$$p_k(\sigma) = p_k(\alpha_k), \quad q_k(\sigma) = 0, \quad \forall \sigma \in [0, \alpha_k] \qquad (10)$$

We shall take into consideration the following sets:

$$\Phi_0 = \{x = \text{col}(\sigma, \eta, z) \mid z \geqslant -{}^1/_2 B\sigma^2\}$$
$$\Phi_k = \{x \mid z \geqslant -P_k(\sigma), \ \eta \leqslant Q_k(\sigma), \ \sigma \in [0, a_k]\}$$
$$\Phi_k^- = \{x \mid z \geqslant -P_k(|\sigma|), \ \eta \geqslant -Q_k(|\sigma|), \ \sigma \in [-a_k, 0]\}$$
$$\psi_k = \{x \mid z \leqslant p_k(\sigma), \ \eta \geqslant q_k(\sigma), \ \sigma \in [0, a_N]\}$$
$$\psi_k^- = \{x \mid z \leqslant p_k(|\sigma|), \ \eta \leqslant -q_k(|\sigma|), \ \sigma \in [-a_N, 0]\}$$
$$\Omega = (\Phi_N \cap \psi_M) \cup (\Phi_N^- \cap \psi_M^-)$$

Theorem 1. Suppose $|q_M(0)| \leqslant \rho$. Then the set Ω is positively invariant. From Theorem 1 it follows that

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Theorem 2. Suppose $\rho = 0, \alpha_M > 0$. Then the separatrix of system (1), which emerges from the saddle $\sigma = \eta = z = 0$, does not vanish as $t \to +\infty$.

The following lemmas are required to prove Theorem 1.

Lemma 1. The set Φ_0 is positively invariant and for any solution $x(t)$ of system (1) the following equation holds:

$$\varliminf_{t \to +\infty} V(t) \geqslant 0 \quad (V(t) = z(t) + {}^1/_2 B\sigma(t)^2) \tag{11}$$

Proof. It is clear that

$$V(t)^{\cdot} = -Az(t) = -AV(t) + {}^1/_2 AB\sigma(t)^2 \geqslant -AV(t)$$

Hence follows the estimate $V(t) \geqslant e^{-At} V(0)$, from which follows the statement of the lemma.

Note that for Lorenz's system, written in classical form

$$x_1^{\cdot} = -\sigma_1(x_1 - y_1), \; y_1^{\cdot} = -x_1z_1 + rx_1 - y_1, \quad z_1^{\cdot} = x_1y_1 - bz_1 \tag{12}$$

inequality (11) will take the form

$$\varliminf_{t \to +\infty} \left(z_1(t) - \frac{1}{2\sigma_1} x_1(t)^2\right) \geqslant 0$$

From this inequality and from V.I. Yudovich's theorem /4, 7/ there follows, in particular, the validity of the hypothesis /8/ that $\varliminf z_1(t) \geqslant 0$ as $t \to +\infty$ and any σ_1, b, r.

Lemma 2. The following inequalities hold:

$$P_{k+1}(\sigma) < P_k(\sigma), \quad Q_{k+1}(\sigma) < Q_k(\sigma), \quad \forall \sigma \in (0, a_{k+1})$$

Proof. From the inequalities $P_1(\sigma) > 0, Q_1(\sigma) > 0, \forall \sigma \in (0, a_1)$ and Eq.(5) it follows that $P_1(\sigma) < P_0(\sigma), \forall \sigma \in (0, a_1)$. Hence using Chaplygin's comparison principle /4/ for Eq.(4) we will obtain that $Q_2(\sigma) < Q_1(\sigma), \forall \sigma \in (0, a_2)$. But then from (5) there follows the inequality $P_2(\sigma) < P_1(\sigma), \forall \sigma \in (0, a_2)$, from which, again using Chaplygin's principle for Eq.(4), we will obtain the estimate $Q_3(\sigma) < Q_2(\sigma), \; \forall \sigma \in (0, a_3)$. Continuing this process further, we obtain the statement of the lemma.

From Lemma 2 there follow the insertions $\Phi_{k+1} \subset \Phi_k, \Phi_{k+1}^- \subset \Phi_k^-$.

Lemma 3. If the relations $x(0) \in \Phi_k, \sigma(t) \geqslant 0, \forall t \in [0, T]$ hold for the solution $x(t)$ of system (1), then $x(t) \in \Phi_k, \forall t \in [0, T]$.

Proof. Bearing in mind the continuous dependence of the solutions on the initial data, it is sufficient to show that the following inequalities hold:

$$\begin{gathered}(z + P_k(\sigma))^{\cdot} > 0, \quad \forall x \in G_1 = \{x \mid \sigma \in (0, a_k), \; z + P_k(\sigma) = 0 \\ \eta < Q_k(\sigma)\} \\ (\eta - Q_k(\sigma))^{\cdot} < 0, \quad \forall x \in G_2 = \{x \mid \sigma \in (0, a_k), \quad z \geqslant -P_k(\sigma) \\ \eta = Q_k(\sigma)\}\end{gathered} \tag{13}$$

We shall first show that the first inequality (13) holds. When $x \in G_1$ we have

$$(z + P_k(\sigma))^{\cdot} = -Az - B\sigma\eta + P_k'\eta = AP_k + (P_k' - B\sigma)\eta = AP_k\left(1 - \frac{\eta}{Q_k(\sigma)}\right) > 0$$

If $x \in G_2$, we have

$$\begin{gathered}(\eta - Q_k(\sigma))^{\cdot} = -\mu\eta - z\sigma - \varphi(\sigma) - Q_k'\eta = -\mu Q_k - Q_k'Q_k - \\ \varphi(\sigma) - z\sigma = -\sigma(P_{k-1}(\sigma) + z) < -\sigma(P_k(\sigma) + z) \leqslant 0\end{gathered}$$

The following lemma is proved in a completely analogous way:

Lemma 4. If the relations $x(0) \in \Phi_k^-, \sigma(t) \leqslant 0, \forall t \in [0, T]$, hold for the solution $x(t)$ of system (1), then $x(t) \in \Phi_k^-, \forall t \in [0, T]$.

Lemma 5. The following inequalities occur:

$$p_{k+1}(\sigma) < p_k(\sigma), \; q_{k+1}(\sigma) > q_k(\sigma), \quad \forall \sigma \in (\alpha_{k+1}, a_N)$$

Proof. From the inequalities $p_1(\sigma) > 0, q_1(\sigma) < 0, \forall \sigma \in (\alpha_1, a_N)$ and Eq.(9) it follows that $p_1(\sigma) < p_0(\sigma), \forall \sigma \in (\alpha_1, a_N)$. Hence, using Chaplygin's principle for Eq.(8), we obtain that $q_2(\sigma) > q_1(\sigma), \forall \sigma \in (\alpha_2, a_N)$. But then from (9) the inequality $p_2(\sigma) < p_1(\sigma), \forall \sigma \in (\alpha_2, a_N)$ follows, from which, again using Chaplygin's principle for Eq.(8), we obtain the estimate $q_3(\sigma) > q_2(\sigma), \forall \sigma \in (\alpha_3, a_N)$. Continuing this process further, we obtain the statement of the lemma.

From Lemma 5 there follow the insertions $\psi_{k+1} \subset \psi_k, \psi_{k+1}^- \subset \psi_k^-$.

Lemma 6. If the relations $x(0) \in \psi_k, \sigma(t) \geqslant 0, \forall t \in [0, T]$ hold for the solution $x(t)$ of system (1), then $x(t) \in \psi_k, \forall t \in [0, T]$.

Proof. It is sufficient to prove that the following inequalities hold:

$$(z - p_k(\sigma))^{\cdot} < 0, \; \forall x \in G_3 = \{x \mid \sigma \in (0, a_N), \quad z = p_k(\sigma), \eta > q_k(\sigma)\} \tag{14}$$

$$(\eta - q_k(\sigma))^{\cdot} > 0, \; \forall x \in G_4 = \{x \mid \sigma \in (0, a_N), \quad z \leqslant p_k(\sigma), \quad \eta = q_k(\sigma)\} \tag{15}$$

For $k = 0$ inequality (14) takes the form $-Az - B\sigma\eta + B\sigma\eta < 0$, i.e. it holds when $x \in G_3$.

When $x \in G_3$ and $\sigma \in (\alpha_k, a_N)$ we have $(z - p_k(\sigma))^{\cdot} = -Az - B\sigma\eta - p_k'\eta = -Ap_k(1 - \eta/q_k(\sigma)) < 0$, and when $x \in G_3$ and $\sigma \in (0, \alpha_k)$ if $\alpha_k > 0$) we will obtain

$$(z - p_k(\sigma))^{\cdot} = -Az - B\sigma\eta < -Ap_k(\alpha_k) < 0$$

Inequality (14) thus holds.

When $x \in G_4$ and $\sigma \in (\alpha_k, a_N)$ we have

$$(\eta - q_k(\sigma))^{\cdot} = -\mu\eta - z\sigma - \varphi(\sigma) - q_k'\eta = -\mu q_k - q_k'q_k - \varphi(\sigma) - z\sigma = \sigma(p_{k-1}(\sigma) - z) > \sigma(p_k(\sigma) - z) \geqslant 0 \tag{16}$$

and when $x \in G_4$ and $\sigma \in (0, \alpha_k)$ (if $\alpha_k > 0$) we will obtain

$$(\eta - q_k(\sigma))^{\cdot} = -\mu\eta - z\sigma - \varphi(\sigma) \geqslant -p_k(\alpha_k)\sigma - \varphi(\sigma) \geqslant \sigma(-p_k(\alpha_k) + 1 - \gamma\sigma^2)$$

We will show that

$$-p_k(\alpha_k) + 1 - \gamma\alpha_k^2 \geqslant 0 \tag{17}$$

Assuming the opposite, we obtain the inequality

$$\eta^{\cdot}(0, x_0) < 0 \tag{18}$$

and from the point

$$x_0 = \operatorname{col}\{\sigma_0, \eta_0, z_0\}, \ \sigma_0 = \alpha_k, \ \eta_0 = 0, \ z_0 = p_k(\alpha_k)$$

we draw the trajectory $x(t, x_0)$. From (18) and the continuous dependence of the solutions of system (1) on the initial data it follows that inequality (16) does not hold when $x \in G_4$, $\sigma \in (\alpha_k, a_N)$ and for fairly small $x - x_0$. This contradiction also proves Eq. (17).

Inequality (15) follows from (16) and (17).

The following lemma is proved in a similar way:

Lemma 7. If the relations $x(0) \in \psi_k^-$, $\sigma(t) \leqslant 0$, $\forall t \in [0, T]$ hold for the solution $x(t)$ of system (1), then $x(t) \in \psi_k^-$, $\forall t \in [0, T]$.

Theorem 1 follows directly from Lemmas 1–7.

Proof of Theorem 2. When $\rho = 0$, $\alpha_M > 0$ the set Ω is divided into two positively invariant sets: $\Omega^+ = \Phi_N \cap \psi_M$ and $\Omega^- = \Phi_N^- \cap \psi_M^-$. It is obvious that in this case the trajectories occurring in Ω^+ and Ω^- cannot approach zero as $t \to +\infty$, since for fairly small σ and $x \in \Omega^+$ we have the inequality $\sigma^{\cdot} = \eta \geqslant 0$, and for fairly small σ and $x \in \Omega^-$ the inequality $\sigma^{\cdot} = \eta \leqslant 0$ holds.

On the other hand, the separatrices of system (1), which emerge from the saddle $x = 0$, are contained either in Ω^+ or in Ω^-. In fact, in some neighbourhood of the point $x = 0$ the separatrix which emerges into the halfspace $\{x \mid \sigma \geqslant 0\}$, coincides with the curve whose equation is $\eta = Q(\sigma)$, $z = -P(\sigma)$. Here $P(\sigma)$ and $Q(\sigma)$ is the solution of system

$$\frac{dQ}{d\sigma} Q + \mu Q + \varphi(\sigma) - P\sigma = 0, \quad \frac{dP}{d\sigma} = B\sigma - \frac{AP}{Q}$$

with the initial data $P(0) = Q(0) = 0$, $Q'(0) > 0$.

It is obvious that $P(\sigma)$ and $Q(\sigma)$ are the limits of the monotonically decreasing (see Lemma 2) sequences $P_k(\sigma)$ and $Q_k(\sigma)$. Hence and from the definition of the set Ω^+ it follows that this separatrix will be contained in some neighbourhood of zero in the set Ω^-. From the positive invariance Ω^+ it then follows that this separatrix will be wholly contained in Ω^-. Arguments can be constructed in a similar way for the separatrix emerging into the halfspace $\{x \mid \sigma \leqslant 0\}$.

Thus, the separatrices leaving the saddle $x = 0$ are contained in Ω^+ and Ω^- and cannot approach zero as $t \to +\infty$.

We shall now use Theorem 2 in the case when $N = M = 2$, $2\gamma > B$. At the same time we will try to obtain, as far as possible, the simplest analytical condition for there to be no separatrix loops, coarsening the above algorithm for finding a_k and α_k.

From Eq. (4) and the inequality $2\gamma > B$ we will find the estimate

$$Q_1(\sigma) \leqslant K_1\sigma, \quad K_1 = -\mu/2 + (1 + \mu^2/4)^{1/2} \tag{19}$$

Using (19), from Eq. (5) we will find the inequality

$$P_1(\sigma) \leqslant C\sigma^2, \quad C = \frac{B}{2 + AK_1^{-1}} \tag{20}$$

From Eq. (4) and estimate (20) it follows that $a_2 < \varkappa$, where $\varkappa$ is a positive root of the equation

$$\int_0^{\varkappa} [\varphi(\sigma) - C\sigma^3]\, d\sigma = 0$$

It is obvious that $\varkappa = [2/(\gamma - C)]^{1/2}$. We shall further assume that the following inequality holds:

$$\mu > 2[(\gamma + C)/(\gamma - C)]^{1/2} \tag{21}$$

From this it follows that a number $K_2 < 0$ exists, for which the following relation holds:

$$K_2^2\sigma + \mu K_2\sigma + \varphi(\sigma) - {}^1/_2 B\sigma^3 + {}^1/_2 B\varkappa^2\sigma < 0, \ \forall \sigma \in (0, \varkappa) \tag{22}$$

This means that $q = K_2\sigma$ is a non-contact straight line of Eq. (8) when $\sigma \in (0, \varkappa)$. Hence it follows that $\alpha_1 = 0$ and

$$q_1(\sigma) \geqslant K_2\sigma, \ \forall \sigma \in [0, a_2] \tag{23}$$

From estimate (23) and Eq.(9) it immediately follows that $p_1(0) = 0$. But then for some fairly small number $\delta > 0$ the inequality $\varphi(\sigma) + p_1(\sigma)\sigma < 0$, $\forall\sigma \in [0, \delta]$ holds. Hence, from inequality (22) and the estimate $p_1(\sigma) < p_0(\sigma)$ it follows that for fairly small δ the following inequality holds:

$$K_2^2(\sigma-\delta) + \mu K_2(\sigma-\delta) + \varphi(\sigma) + p_1(\sigma)\sigma < 0, \ \forall\ \sigma \in (\delta, \alpha_2)$$

This means that $q = K_2(\sigma - \delta)$ is a non-contact straight line for Eq.(8) when $\sigma \in (\delta, \alpha_2)$. But then $\alpha_2 \geqslant \delta > 0$.

Thus, for the conditions of Theorem 2 to hold, it is sufficient that inequality (21) holds.

For Lorenz's system, written in the standard form (12), this condition will take the form

$$\frac{(\sigma_1+1)^2}{(r-1)\sigma_1} > 4\frac{\sigma_1+\Lambda}{\sigma_1-\Lambda}, \quad \Lambda = \frac{2\sigma_1 - b}{2 + b(\sigma_1+1)\sigma_1^{-1}(r-1)^{-1}} \tag{24}$$

It is obvious that condition (24) holds when $\sigma_1 = 10$, $b = 8/3$, $r \leqslant 2$. For large values of σ_1, r estimate (24) takes the form $r < 1/4\,(b\sigma_1)^{1/2}$.

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ON THE SINGULARITY OF THE STRESSES NEAR THE FACE OF A THIN ELASTIC INCLUSION IN A PIECEWISE HOMOGENEOUS PLANE*

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The asymptotic behaviour of normal stresses near the tip of a thin elastic inclusion situated near a line weld joining two dissimilar elastic half-planes, is studied. It is established that apart from the well-known root-type singularity /1/ two additional terms of the asymptotic expression exist which must not be neglected. One of them is of the order of unity, and the other contains an "imaginary singularity" and makes a singificant contribution to the state of stress when the distances between the face of the inclusion and the line separating the materials are small.

1. Normal stresses and their asymptotic behaviour. A thin elastic inclusion of normalized length 2 (here and henceforth all distances will be expressed in terms of the half-length of the inclusion), is situated in one of the welded isotropic half-planes possessing different elastic characteristics. The distance between the right end of the inclusion and the line separating the materials is δ (Fig.1). A field of tensile stresses σ_1 and σ_2 exists at a sufficient distance from the inclusion, and we have $\sigma_2 = \sigma_1(1+\varkappa_1)\mu_2/[(1+\varkappa_2)\mu_1]$, $\varkappa_j = (3-\nu_j)/(1+\nu_j)$ for the generalized plane stress state, $\varkappa_j = 3-4\nu_j$ for plane stress, μ_j is the shear modulus and ν_j is Poisson's ratio of the materials of the half-planes $(j = 1, 2)$. The corresponding quantities with zero index refer to the material of the inclusion.

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